Stable Roommates for Weighted Straight Skeletons

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Abstract

In this paper, we fill in a gap in the wavefront-based definition of weighted straight skeletons in the presence of multiple simultaneous, co-located split events. We interpret the need to pair up wavefront edges to restore planarity as a particular matching problem. Our results on a stable roommate problem defined on a directed pseudo-line arrangement show that our method always yields a solution. We thereby complete the definition of weighted straight skeletons.

1 Introduction

The straight skeleton is a skeletal structure similar to the medial axis. It was introduced to computational geometry almost two decades ago by Aichholzer et al. [1]. The definition of the straight skeleton $\mathcal{S}(P)$ of a polygon P, possibly with holes, is based on a wavefront sent out by P which forms a mitered offset $\mathcal{W}_P(t)$ of P at any given time $t \geq 0$. The wavefront undergoes two different kinds of topological changes (so-called events) over time: Roughly speaking, an *edge event* happens when a wavefront edge collapses and a *split event* happens when the wavefront splits into parts. The straight skeleton $\mathcal{S}(P)$ of P is defined as the geometric graph whose edges are the traces of the vertices of $\mathcal{W}_P$, see Fig. 1. Similar to Voronoi diagrams and the medial axis, straight skeletons became a versatile tool for applications in various domains of science and industry [7].

Eppstein and Erickson [5] were the first to mention *weighted straight skeletons*, where wavefront edges do not necessarily move with unit speed. Weighted straight skeletons are used in different applications [2, 6, 8, 9] but also constitute a theoretical tool to generalize straight skeletons to 3D [3]. Even though weighted straight skeletons have already been applied in both theory and practice, only recently Biedl et al. [4] showed that basic properties of ordinary straight skeletons do not carry over to weighted straight skeletons in general. Biedl et al. [4] also proposed solutions for an ambiguity in the definition of straight skeletons caused by certain edge events and first mentioned by Kelly and Wonka [8] and Huber [7].

In this paper, we discuss another open problem in the definition of weighted straight skeletons caused by split events. An event happens due to a topological change in the wavefront and the event handling is guided by one fundamental principle: *Between events, the wavefront is a planar*[1] *collection of wavefront polygons.* This is easily achieved when handling edge events and "simple" split events. However, is it always possible to handle multiple simultaneous, co-located split events in a fashion such that the wavefront remains planar?

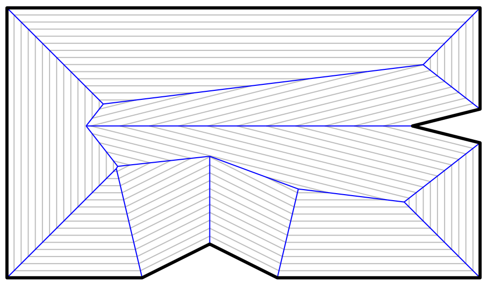

Figure 1: The straight skeleton $\mathcal{S}(P)$ (blue) of the input polygon P (bold) is defined by the wavefront $\mathcal{W}_P$ (gray) emanated from P.

We can answer this question to the affirmative and therefore show how to define weighted straight skeletons safely in the presence of multiple simultaneous, co-located split events. (Note that due to the discontinuous character of straight skeletons, it is not possible to tackle this problem by means of simulation of simplicity.) We first rephrase this problem as a so-called *planar matching* problem of directed pseudo-lines and show how to transform the planar matching problem into a *stable roommate* problem. For the main result, we prove that our particular stable roommate problem at hand always possesses a solution and those solutions tell us how to do the event handling of the wavefront in order to maintain planarity.

2 Weighted straight skeletons

The wavefront. For our further discussions it will be necessary to define precisely what we mean by edge

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[1] By "planar" we mean that the embedding is planar, i.e. only adjacent edges intersect at their common endpoint.

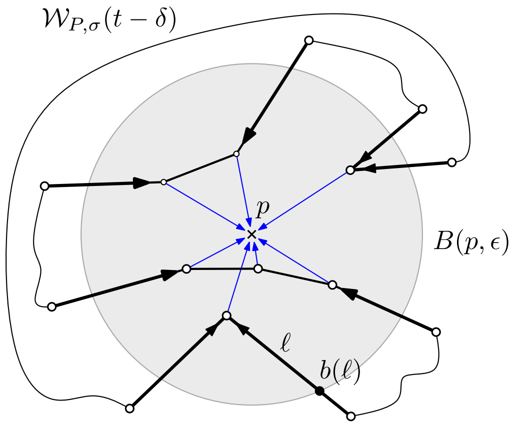

Figure 2: A multi-split event occurs at location p. The involved edges form 4 chains. Exactly 8 involved edges do not collapse (arrows).

and split events. Let P denote a polygon with holes. Following the notation by Biedl et al. [4], we denote by $\sigma(e) \in \mathbb{R} \setminus \{0\}$ the *weight* of the edge e of P and call σ the *weight function*. For every edge e of P let $n(e)$ denote the normal vector of e that points to the interior of P. Initially, every edge of P sends out a wavefront edge with fixed speed $\sigma(e)$. That is, the segments of the wavefront $\mathcal{W}_{P,\sigma}(t)$ at time t that originate from the polygon edge e are contained in $\overline{e} + t \cdot \sigma(e) \cdot n(e)$, where $\overline{e}$ denotes the supporting line of e. If $\sigma(e)$ is negative, the wavefront edge emanated from e moves to the exterior of P.

Events. Intuitively, an event happens when a wavefront vertex meets another wavefront edge or, in particular, another wavefront vertex. The situation becomes more complicated when two or more such events are co-located at the same time t. Assume that $\mathcal{W}_{P,\sigma}(t')$ remains planar for all $t - \delta_0 < t' < t$ and $\delta_0 > 0$. In particular, no three edges occupy a common locus. For this time interval, we can consider $\mathcal{W}_{P,\sigma}$ to be a kinetic planar straight-line graph (PSLG) with a fixed set of kinetic vertices and edges and we denote by $e(t')$ the segment that is occupied by edge e of $\mathcal{W}_{P,\sigma}$ at time t'. Furthermore, we denote by $B(p, r)$ the closed disk centered at p with radius r.

Definition 1 *Let $e_1, \dots, e_n$ be the edges of the wavefront $\mathcal{W}_{P,\sigma}(t-\delta)$ for a sufficiently small $\delta > 0$. At location p at time t an* event *happens if there are indices $\{i_1, \dots, i_k\}$, with $k \geq 3$, such that*

$$\exists \varepsilon_0 > 0\ \forall 0 < \varepsilon < \varepsilon_0\ \exists \delta_0 > 0\ \forall 0 < \delta < \delta_0 :$$
$$e_j(t-\delta) \cap B(p,\varepsilon) \neq \emptyset \ \Leftrightarrow j \in \{i_1, \dots, i_k\}.$$

We call $e_{i_1}, \dots, e_{i_k}$ the edges that are involved *in the event.*

In other words, an event happens when for each ball of a sequence of shrinking balls around p there exists an interval ending in time t such that the set of edges intersecting the ball throughout that interval is constant and the same for all balls.

If an event happens at location p and time t then planarity of $\mathcal{W}_{P,\sigma}(t)$ is violated locally at p as all involved edges meet at p. The goal of the event handling is to restore planarity locally. We can classify events as follows.

Definition 2 *We call the event at location p and time t* elementary *if three edges are involved. We call it an* edge event *if $B(p,\varepsilon) \setminus \mathcal{W}_{P,\sigma}(t-\delta)$ consists of two connected components and a* split event *otherwise. Non-elementary edge and split events are called multi-edge and multi-split events respectively.*

It is known how to handle edge events and elementary split events [4]. In the following, we will show how to handle multi-split events in a way such that planarity is locally restored.

Pairing edges. Suppose a multi-split event happens at location p and time t. Note that $\mathcal{W}_{P,\sigma}(t-\delta) \cap B(p,\varepsilon)$ consists of k polygonal chains, see Fig. 2. For every involved edge either zero, one, or both endpoints approach p as time goes to t. In latter case, the involved edge collapses to length zero and is removed. In the first case, we simplify the further discussion by splitting the edge by a wavefront vertex within $B(p,\epsilon)$ that reaches p at time t. Now we have precisely $2k$ wavefront edges that remain and which have exactly one endpoint that reaches p at time t.

The goal is now to find a pairing for these $2k$ edges such that (i) we reconnect a pair of edges by a new wavefront vertex and (ii) the resulting wavefront is planar again after time t locally at p. In the following, we will rephrase the problem of finding such a pairing as a particular matching problem. For the sake of simplicity, we assume for the supporting lines of the $2k$ edges that (i) every pair intersects in exactly one locus and (ii) no triple intersects (except for isolated points in time).[2]

3 Matchings and roommates

Planar matchings. In this section, assume we keep propagating the wavefront without change after a multi-split event, resulting in non-planarity of the wavefront at time $t + \delta$ around p. We choose ϵ_0, δ_0 small enough such that for all ϵ, δ with $0 < \epsilon < \epsilon_0$ and

[2]In order to achieve these requirements, we can think of rotationally perturbing the wavefront edges in a specific way. Undoing the perturbation may destroy planarity in the strict sense, but yields planarity in a weaker sense where wavefront edges may touch. We can also show that for some input such "touching" cannot be avoided. Due to space limitations we cannot go into details here.

$0 < \delta < \delta_0$ no non-involved edge of $\mathcal{W}_{P,\sigma}(t+\delta)$ intersects $B(p,\epsilon)$, the intersection points among the supporting lines of involved edges are contained within $B(p,\epsilon)$, and no triple of supporting lines intersected again. We denote by $l_1,\ldots,l_N$ the line segments occupied by the $N = 2k$ non-collapsed involved edges at time $t+\delta$. We can choose ϵ_0 small enough such that exactly one endpoint of each l_i is contained in $B(p,\epsilon)$. Hence, we can orient $l_1,\ldots,l_N$ such that they point inside $B(p,\epsilon)$, see Fig. 2. Extending all l_i to their supporting lines $\overline{l_i}$ gives us an arrangement of N directed lines where each pair intersects in a unique locus.

We generalize this arrangement to a pseudo-line arrangement $\mathcal{L}$ of N directed pseudo-lines $\ell_1,\ldots,\ell_N$ whose intersections are contained in $B(p,\epsilon)$. We assume that no pseudo-line intersects itself, every pair of pseudo-lines intersects exactly once, no three pseudo-lines intersect in a point, every directed pseudo-line begins and ends at infinity and intersects the boundary $\operatorname{bd} B(p,\epsilon)$ exactly twice. The direction of a pseudo-line ℓ allows us to distinguish between the *begin-point* $b(\ell)$, at its first intersection with $\operatorname{bd} B(p,\epsilon)$, and the end-point, at its last intersection with $\operatorname{bd} B(p,\epsilon)$.

A *matching* in $\mathcal{L}$ is a grouping of $\ell_1,\ldots,\ell_N$ into pairs of matching partners. The *matching tail* of ℓ_i is the part of ℓ_i from $b(\ell_i)$ to the intersection point with its matching partner.

Definition 3 *A matching in $\mathcal{L}$ is* planar *if no two matching tails intersect except at their ends.*

The matching tails play the role of the wavefront edges after the event. The matching tells us how to pair up the wavefront edges in order to restore planarity locally at p.

Lemma 1 *There exists a planar offset after the event if and only if there is a planar matching for $\mathcal{L}$.*

The solutions of the planar offset problem and the planar matching problem are in a one-to-one correspondence and, thus, the planar offset problem may have multiple solutions. The main result of this paper is the following theorem, which says that at least one solution always exists.

Theorem 2 *Every directed pseudo-line arrangement has a planar matching.*

The stable roommate problem. Assume we have an even number N of elements $a_1,\ldots,a_N$. Each element has a ranking (i.e., preference list) of elements. The rankings are *complete* (all elements are contained) and *strict* (no two elements are ranked the same). Let M be a matching of $a_1,\ldots,a_N$ and let $M(a_i)$ denote the matching partner of a_i. A pair $\{a_i,a_j\}$ is a *blocking pair* if a_i prefers a_j over $M(a_i)$ and a_j prefers a_i over $M(a_j)$. A matching is *stable* if there is no blocking pair. The *stable roommate problem* asks for a stable matching for $a_1,\ldots,a_N$ and their rankings. The stable roommate problem is a well studied problem in optimization and it is well known that not every instance of the stable roommate problem has a solution.

Let us again consider the pseudo-line arrangement $\mathcal{L}$. As we walk along a pseudo-line ℓ_i from its begin-point to its end-point we encounter all other pseudo-lines in $\mathcal{L}$. This order naturally gives us a complete and strict ranking for ℓ_i if we attach ℓ_i itself at the end of the list. Thus, $\mathcal{L}$ defines an instance of the stable roommate problem. The following lemma translates the planar matching problem to the corresponding stable roommate problem.

Lemma 3 *A directed pseudo-line arrangement has a planar matching if and only if the corresponding stable roommate instance has a stable matching.*

Proof. Let M be a matching. The matching tails of two pseudo-lines ℓ_i,ℓ_j cross if and only if ℓ_i prefers ℓ_j over $M(\ell_i)$ and ℓ_j prefers ℓ_i over $M(\ell_j)$. Hence, the matching is non-planar if and only if there is a blocking pair. ☐

In light of this lemma, we will prove Thm. 2 by showing that the stable roommate problem defined by $\mathcal{L}$ has a solution. In order to do so, we need to review a few results concerning stable roommate problems, mostly based on the paper by Tan and Hsueh [10].

Stable partitions. Let A be a set of an even number N of elements $a_1,\ldots,a_N$, together with complete and strict preference lists, and let π be a bijective map $A \to A$. This function partitions A into one or more *cycles*, i.e., sequences $a'_0,\ldots,a'_{k-1}$ in A with $a'_0 \xrightarrow{\pi} a'_1 \xrightarrow{\pi} \ldots \xrightarrow{\pi} a'_{k-1} \xrightarrow{\pi} a'_0$. A cycle with $k \geq 3$ is called a *semi-party cycle* if a'_i prefers $\pi(a'_i)$ over $\pi^{-1}(a'_i)$. A *semi-party partition* is a permutation where all cycles with $k \geq 3$ are semi-party cycles.

Given a semi-party partition π, a pair $\{a_i,a_j\}$ is called a *party-blocking pair* if a_i prefers a_j over $\pi^{-1}(a_i)$ and a_j prefers a_i over $\pi^{-1}(a_j)$. A *stable partition* is a semi-party partition that has no party-blocking pairs. The cycles of a stable partition are called *parties*. An odd (even) party is a party of odd (even) cardinality. Furthermore, a_i,a_j are *party partners* if $a_i = \pi(a_j)$ or $a_j = \pi(a_i)$.

Theorem 4 ([10]) *For any instance $\mathcal{A}$ of the stable roommate problem the following statements hold:*

1. *$\mathcal{A}$ has a stable partition and it can be found in polynomial time.*
2. *$\mathcal{A}$ has a stable matching if and only if it has a stable partition with no odd parties.*
3. *If $\mathcal{A}$ has a stable matching, then it can be found in polynomial time.*

No odd parties in $\mathcal{L}$. Thm. 4(1) gives us a stable partition π for $\mathcal{L}$. Let a *singleton-party*, a *pair-party*, and a *cycle-party* be a party consisting of one, two, and at least three pseudo-lines, respectively. For two pseudo-lines ℓ, ℓ' let $\ell \times \ell'$ be the point of intersection. For all pseudo-lines ℓ posing a singleton-party, let its *tail* be the part of ℓ between begin- and end-point. For all other pseudo-lines ℓ let their *tail* be the part between $b(\ell)$ and $\ell \times \pi^{-1}(\ell)$.

Lemma 5 *The tails of two pseudo-lines ℓ, ℓ' do not intersect unless ℓ and ℓ' are party-partners.*

Proof. Assume that $\ell \times \ell'$ belongs to both tails, but ℓ and ℓ' are no party-partners. Then ℓ prefers ℓ' over $\pi^{-1}(\ell)$ and ℓ' prefers ℓ over $\pi^{-1}(\ell')$. Hence, $\{\ell, \ell'\}$ is a party-blocking pair and π is not a stable partition. □

Lemma 6 *There cannot be two cycle-parties.*

Proof. Assume we have two cycle-parties P_1 and P_2. We consider the graph G whose vertex set comprises $b(\ell)$ and $\ell \times \pi(\ell)$ for every pseudo-line ℓ in $P_1 \cup P_2$. We connect all $b(\ell)$ cyclically by edges drawn on $\operatorname{bd} B(p, \epsilon)$. We also add edges from $b(\ell)$ to $\ell \times \pi(\ell)$ and from $\ell \times \pi(\ell)$ to $\pi^{-1}(\ell) \times \ell$, both drawn on ℓ. By Lem. 5 the graph G is planar.

First of all, G contains a path $b(\ell) \to \ell \times \pi(\ell) \to \pi^{-1}(\ell) \times \ell \leftarrow b(\pi^{-1}(\ell))$ for any ℓ in $P_1 \cup P_2$. Secondly, the sequence $b(\ell), b(\pi(\ell)), b(\pi(\pi(\ell))), \ldots$ of begin-points of a single party appear in cyclic order on $\operatorname{bd} B(p, \epsilon)$. Hence, we can renumber all pseudo-lines $\ell_1, \ldots, \ell_m$ in $P_1 \cup P_2$ such that (i) $b(\ell_1), \ldots, b(\ell_m)$ appear cyclically on $B(p, \epsilon)$ and (ii) $\ell_1, \ldots, \ell_k$ are from P_1 and $\ell_{k+1}, \ldots, \ell_m$ are from P_2.

Let the P_1-region be the region of $B(p, \epsilon)$ enclosed by the cycle formed by $b(\ell_1), \ldots, b(\ell_k)$ and the 3-path connecting $b(\ell_k)$ and $b(\ell_1)$. Likewise, the P_2-region is given by $b(\ell_{k+1}), \ldots, b(\ell_m)$ and the 3-path between $b(\ell_m)$ and $b(\ell_{k+1})$. Note that the P_1-region and the P_2-region are disjoint.

As ℓ_1 and ℓ_m belong to $\mathcal{L}$, they need to intersect within $B(p, \epsilon)$. But ℓ_1 starts at $b(\ell_1)$, lives on the boundary of the P_1-region until $\ell_1 \times \ell_k$, and then enters the P_1 region. From there it cannot leave the P_1-region without intersecting ℓ_1 itself or ℓ_k a second time or by leaving $B(p, \epsilon)$. Once it left $B(p, \epsilon)$, it cannot reenter again. Similarly, ℓ_2 is bound to the P_2-region within $B(p, \epsilon)$. As the P_1-region and the P_2-region are disjoint, ℓ_1 and ℓ_m have no intersection, which is a contradiction. □

A similar argument shows the following lemma:

Lemma 7 *There cannot be a singleton-party and a cycle-party.*

The following theorem finally proves Thm. 2 by Lemma 3 and Thm. 4(1-2).

Theorem 8 *No instance of a stable roommate problem defined by a directed pseudo-line arrangement $\mathcal{L}$ can have an odd party.*

Proof. Assume to the contrary that there is an odd party P. As $\mathcal{L}$ comprises an even number of pseudo-lines, there needs to be another odd party P'. Lem. 6 and Lem. 7 imply that P and P' need to be both singleton-parties. Since elements prefer themselves least, P and P' constitute a party-blocking pair. □

By Thm. 4(3) we can hence find a stable matching, and with it a planar post-event wavefront, in polynomial time.

Acknowledgments

We would like to thank David Eppstein for mentioning this problem to us, and for the idea of interpreting the edge-pairing problem as a stable roommate problem.

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